NOTE OF ELEMENTARY ANALYSIS II

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1. Riemann Integrals

Notation 1.1. .

(i) : *All functions $f, g, h...$ are bounded real valued functions defined on $[a, b]$. And $m \leq f \leq M$.*
(ii) : *$\mathcal{P} : a = x_0 < x_1 < .... < x_n = b$ denotes a partition on $[a, b]$; $\Delta x_i = x_i - x_{i-1}$ and $\|\mathcal{P}\| = \max \Delta x_i$.*
(iii) : *$M_i(f, \mathcal{P}) := \sup\{f(x) : x \in [x_{i-1}, x_i\}$; $m_i(f, \mathcal{P}) := \inf\{f(x) : x \in [x_{i-1}, x_i\}$. And $\omega_i(f, \mathcal{P}) = M_i(f, \mathcal{P}) - m_i(f, \mathcal{P})$.*
(iv) : *$U(f, \mathcal{P}) := \sum M_i(f, \mathcal{P})\Delta x_i$; $L(f, \mathcal{P}) := \sum m_i(f, \mathcal{P})\Delta x_i$.*
(v) : *$\mathcal{R}(f, \mathcal{P}, \{\xi_i\}) := \sum f(\xi_i)\Delta x_i$, where $\xi_i \in [x_{i-1}, x_i]$.*
(vi) : *$\mathcal{R}[a, b]$ is the class of all Riemann integral functions on $[a, b]$.*

Definition 1.2. *We say that the Riemann sum $\mathcal{R}(f, \mathcal{P}, \{\xi_i\})$ converges to a number A as $\|\mathcal{P}\| \to 0$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that*

$$|A - \mathcal{R}(f, \mathcal{P}, \{\xi_i\})| < \varepsilon$$

for any $\xi_i \in [x_{i-1}, x_i]$ whenever $\|\mathcal{P}\| < \delta$.

Theorem 1.3. *$f \in \mathcal{R}[a, b]$ if and only if for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.*

Lemma 1.4. *$f \in \mathcal{R}[a, b]$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$ whenever $\|\mathcal{P}\| < \delta$.*

Proof. The converse follows from Theorem 1.3.
Assume that f is integrable over $[a, b]$. Let $\varepsilon > 0$. Then there is a partition $\mathcal{Q} : a = y_0 < ... < y_l = b$ on $[a, b]$ such that $U(f, Q) - L(f, Q) < \varepsilon$. Now take $0 < \delta < \varepsilon/l$. Suppose that $\mathcal{P} : a = x_0 < ... < x_n = b$ with $\|\mathcal{P}\| < \delta$. Then we have

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = I + II$$

where

$$I = \sum_{i:Q\cap(x_{i-1},x_i)=\emptyset} \omega_i(f, \mathcal{P})\Delta x_i;$$

and

$$II = \sum_{i:Q\cap(x_{i-1},x_i)\neq\emptyset} \omega_i(f, \mathcal{P})\Delta x_i$$

Notice that we have

$$I \leq U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < \varepsilon$$

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and

$$II \leq (M-m) \sum_{i:Q\cap(x_{i-1},x_i)\neq\emptyset} \Delta x_i \leq (M-m)\cdot l \cdot \frac{\varepsilon}{l} = (M-m)\varepsilon.$$

The proof is finished. □

Theorem 1.5. *$f \in \mathcal{R}[a,b]$ if and only if the Riemann sum $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ is convergent. In this case, $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\int_a^b f(x)dx$ as $\|\mathcal{P}\| \to 0$.*

Proof. For the proof $(\Rightarrow)$: we first note that we always have

$$L(f,\mathcal{P}) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) \leq U(f,\mathcal{P})$$

and

$$L(f,\mathcal{P}) \leq \int_a^b f(x)dx \leq U(f,\mathcal{P})$$

for any $\xi_i \in [x_{i-1},x_i]$ and for all partition $\mathcal{P}$.

Now let $\varepsilon > 0$. Lemma 1.4 gives $\delta > 0$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$ as $\|\mathcal{P}\| < \delta$. Then we have

$$|\int_a^b f(x)dx - \mathcal{R}(f,\mathcal{P},\{\xi_i\})| < \varepsilon$$

as $\|\mathcal{P}\| < \delta$. The necessary part is proved and $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\int_a^b f(x)dx$.

For $(\Leftarrow)$: there exists a number A such that for any $\varepsilon > 0$, there is $\delta > 0$, we have

$$A - \varepsilon < \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon$$

for any partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ and $\xi_i \in [x_{i-1},x_i]$.

Now fix a partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$. Then for each $[x_{i-1},x_i]$, choose $\xi_i \in [x_{i-1},x_i]$ such that $M_i(f,\mathcal{P}) - \varepsilon \leq f(\xi_i)$. This implies that we have

$$U(f,\mathcal{P}) - \varepsilon(b-a) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon.$$

So we have shown that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$\overline{\int_a^b} f(x)dx \leq U(f,\mathcal{P}) \leq A + \varepsilon(1+b-a). \tag{1.1}$$

By considering $-f$, note that the Riemann sum of $-f$ will converge to $-A$. The inequality 1.1 will imply that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$A - \varepsilon(1+b-a) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq A + \varepsilon(1+b-a).$$

The proof is finished. □

Theorem 1.6. *Let $f \in \mathcal{R}[c,d]$ and let $\phi : [a,b] \longrightarrow [c,d]$ be a strictly increasing C^1 function with $f(a) = c$ and $f(b) = d$.*
Then $f \circ \phi \in \mathcal{R}[a,b]$, moreover, we have

$$\int_c^d f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt.$$

Proof. Let $A = \int_c^d f(x)dx$. By Theorem 1.5, we need to show that for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| < \varepsilon$$

for all $\xi_k \in [t_{k-1}, t_k]$ whenever $\mathcal{Q} : a = t_0 < ... < t_m = b$ with $\|\mathcal{Q}\| < \delta$.
Now let $\varepsilon > 0$. Then by Lemma 1.4 and Theorem 1.5, there is $\delta_1 > 0$ such that

$$|A - \sum f(\eta_k)\triangle x_k| < \varepsilon \tag{1.2}$$

and

$$\sum \omega_k(f, \mathcal{P})\triangle x_k < \varepsilon \tag{1.3}$$

for all $\eta_k \in [x_{k-1}, x_k]$ whenever $\mathcal{P} : c = x_0 < ... < x_m = d$ with $\|\mathcal{P}\| < \delta_1$.
Now put $x = \phi(t)$ for $t \in [a, b]$.
Now since ϕ and ϕ' are continuous on $[a, b]$, there is $\delta > 0$ such that $|\phi(t) - \phi(t')| < \delta_1$ and $|\phi'(t) - \phi'(t')| < \varepsilon$ for all $t, t'\ in [a, b]$ with $|t - t'| < \delta$.
Now let $\mathcal{Q} : a = t_0 < ... < t_m = b$ with $\|\mathcal{Q}\| < \delta$. If we put $x_k = \phi(t_k)$, then $\mathcal{P} : c = x_0 < .... < x_m = d$ is a partition on $[c, d]$ with $\|\mathcal{P}\| < \delta_1$ because ϕ is strictly increasing.
Note that the Mean Value Theorem implies that for each $[t_{k-1}, t_k]$, there is $\xi_k^* \in (t_{k-1}, t_k)$ such that

$$\triangle x_k = \phi(t_k) - \phi(t_{k-1}) = \phi'(\xi_k^*)\triangle t_k.$$

This yields that

$$|\triangle x_k - \phi'(\xi_k)\triangle t_k| < \varepsilon\triangle t_k \tag{1.4}$$

for any $\xi_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., m$ because of the choice of δ.
Now for any $\xi_k \in [t_{k-1}, t_k]$, we have

$$\begin{aligned} |A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| &\leq |A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| \\ &+ |\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \\ &+ |\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \end{aligned} \tag{1.5}$$

Notice that inequality 1.2 implies that

$$|A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| = |A - \sum f(\phi(\xi_k^*))\triangle x_k| < \varepsilon.$$

Also, since we have $|\phi'(\xi_k^*) - \phi'(\xi_k)| < \varepsilon$ for all $k = 1, .., m$, we have

$$|\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \leq M(b - a)\varepsilon$$

where $|f(x)| \leq M$ for all $x \in [c, d]$.
On the other hand, by using inequality 1.4 we have

$$|\phi'(\xi_k)\triangle t_k| \leq \triangle x_k + \varepsilon\triangle t_k$$

for all k. This, together with inequality 1.3 imply that

$$\begin{aligned} &|\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \\ &\leq \sum \omega_k(f, \mathcal{P})|\phi'(\xi_k)\triangle t_k| \ (\because \phi(\xi_k^*), \phi(\xi_k) \in [x_{k-1}, x_k]) \\ &\leq \sum \omega_k(f, \mathcal{P})(\triangle x_k + \varepsilon\triangle t_k) \\ &\leq \varepsilon + 2M(b - a)\varepsilon. \end{aligned}$$

Finally by inequality 1.5, we have

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \leq \varepsilon + M(b-a)\varepsilon + \varepsilon + 2M(b-a)\varepsilon.$$

The proof is finished. □

Example 1.7. *Define (*formally*) an improper integral $\Gamma(s)$ (called the Γ-function) as follows:*

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx$$

for $s \in \mathbb{R}$. Then $\Gamma(s)$ is convergent if and only if $s > 0$.

Proof. Put $I(s) := \int_0^1 x^{s-1}e^{-x}dx$ and $II(s) := \int_1^\infty x^{s-1}e^{-x}dx$. We first claim that the integral $II(s)$ is convergent for all $s \in \mathbb{R}$.
In fact, if we fix $s \in \mathbb{R}$, then we have

$$\lim_{x\to\infty} \frac{x^{s-1}}{e^{x/2}} = 0.$$

So there is $M > 1$ such that $\frac{x^{s-1}}{e^{x/2}} \leq 1$ for all $x \geq M$. Thus we have

$$0 \leq \int_M^\infty x^{s-1}e^{-x}dx \leq \int_M^\infty e^{-x/2}dx < \infty.$$

Therefore we need to show that the integral $I(s)$ is convergent if and only if $s > 0$.
Note that for $0 < \eta < 1$, we have

$$0 \leq \int_\eta^1 x^{s-1}e^{-x}dx \leq \int_\eta^1 x^{s-1}dx = \begin{cases} \frac{1}{s}(1-\eta^s) & \text{if } s-1 \neq -1; \\ -\ln\eta & \text{otherwise .} \end{cases}$$

Thus the integral $I(s) = \lim\limits_{\eta\to 0+} \displaystyle\int_\eta^1 x^{s-1}e^{-x}dx$ is convergent if $s > 0$.
Conversely, we also have

$$\int_\eta^1 x^{s-1}e^{-x}dx \geq e^{-1}\int_\eta^1 x^{s-1}dx = \begin{cases} \frac{e^{-1}}{s}(1-\eta^s) & \text{if } s-1 \neq -1; \\ -e^{-1}\ln\eta & \text{otherwise .} \end{cases}$$

So if $s \leq 0$, then $\int_\eta^1 x^{s-1}e^{-x}dx$ is divergent as $\eta \to 0+$. The result follows. □

2. Uniform Convergence of a Sequence of Differentiable Functions

Proposition 2.1. *Let $f_n : (a,b) \longrightarrow \mathbb{R}$ be a sequence of functions. Assume that it satisfies the following conditions:*

(i) : $f_n(x)$ point-wise converges to a function $f(x)$ on (a,b);
(ii) : each f_n is a C^1 function on (a,b);
(iii) : $f_n' \to g$ uniformly on (a,b).

Then f is a C^1-function on (a,b) with $f' = g$.

Proof. Fix $c \in (a,b)$. Then for each x with $c < x < b$ (similarly, we can prove it in the same way as $a < x < c$), the Fundamental Theorem of Calculus implies that

$$f_n(x) = \int_c^x f'(t)dt.$$

Since $f_n' \to g$ uniformly on (a,b), we see that

$$\int_c^x f_n'(t)dt \longrightarrow \int_c^x g(t)dt.$$

This gives

$$f(x) = \int_c^x g(t)dt. \tag{2.1}$$

for all $x \in (c,b)$. On the other hand, g is continuous on (a,b) since each f_n' is continuous and $f_n' \to g$ uniformly on (a,b). Equation 2.1 will tell us that f' exists and $f' = g$ on (c,b). The proof is finished. □

Proposition 2.2. *Let (f_n) be a sequence of differentiable functions defined on (a,b). Assume that*

(i): there is a point $c \in (a,b)$ such that $\lim f_n(c)$ exists;
(ii): f_n' converges uniformly to a function g on (a,b).

Then

(a): f_n converges uniformly to a function f on (a,b);
(b): f is differentiable on (a,b) and $f' = g$.

Proof. For Part (a), we will make use the Cauchy theorem.
Let $\varepsilon > 0$. Then by the assumptions (i) and (ii), there is a positive integer N such that

$$|f_m(c) - f_n(c)| < \varepsilon \quad \text{and} \quad |f_m'(x) - f_n'(x)| < \varepsilon$$

for all $m, n \geq N$ and for all $x \in (a,b)$. Now fix $c < x < b$ and $m, n \geq N$. To apply the Mean Value Theorem for $f_m - f_n$ on (c,x), then there is a point ξ between c and x such that

$$f_m(x) - f_n(x) = f_m(c) - f_n(c) + (f_m'(\xi) - f_n'(\xi))(x - c). \tag{2.2}$$

This implies that

$$|f_m(x) - f_n(x)| \leq |f_m(c) - f_n(c)| + |f_m'(\xi) - f_n'(\xi)||x - c| < \varepsilon + (b-a)\varepsilon$$

for all $m, n \geq N$ and for all $x \in (c,b)$. Similarly, when $x \in (a,c)$, we also have

$$|f_m(x) - f_n(x)| < \varepsilon + (b-a)\varepsilon.$$

So Part (a) follows. Let f be the uniform limit of (f_n) on (a,b)

For Part (b), we fix $u \in (a,b)$. We are going to show

$$\lim_{x \to u} \frac{f(x) - f(u)}{x - u} = g(u).$$

Let $\varepsilon > 0$. Since $f_n \to f$ and $f' \to g$ both are uniformly convergent on (a,b). Then there is $N \in \mathbb{N}$ such that

$$|f_m(x) - f_n(x)| < \varepsilon \quad \text{and} \quad |f_m'(x) - f_n'(x)| < \varepsilon \tag{2.3}$$

for all $m, n \geq N$ and for all $x \in (a,b)$
Note that for all $m \geq N$ and $x \in (a,b) \setminus \{u\}$, applying the Mean value Theorem for $f_m - f_N$ as before, we have

$$\frac{f_m(x) - f_N(x)}{x - u} = \frac{f_m(u) - f_N(u)}{x - u} + (f_m'(\xi) - f_N'(\xi))$$

for some ξ between u and x.
So Eq.2.3 implies that

$$|\frac{f_m(x)-f_m(u)}{x-u}-\frac{f_N(x)-f_N(u)}{x-u}|\leq\varepsilon \tag{2.4}$$

for all $m\geq N$ and for all $x\in(a,b)$ with $x\neq u$.
Taking $m\to\infty$ in Eq.2.4, we have

$$|\frac{f(x)-f(u)}{x-u}-\frac{f_N(x)-f_N(u)}{x-u}|\leq\varepsilon.$$

Hence we have

$$\begin{aligned}|\frac{f(x)-f(u)}{x-u}-f_N'(u)|&\leq|\frac{f(x)-f(u)}{x-c}-\frac{f_N(x)-f_N(u)}{x-u}|+|\frac{f_N(x)-f_N(u)}{x-u}-f_N'(u)|\\&\leq\varepsilon+|\frac{f_N(x)-f_N(u)}{x-u}-f_N'(u)|.\end{aligned}$$

So if we can take $\delta>0$ such that $|\frac{f_N(x)-f_N(u)}{x-u}-f_N'(u)|<\varepsilon$ for $0<|x-u|<\delta$, then we have

$$|\frac{f(x)-f(u)}{x-u}-f_N'(u)|\leq 2\varepsilon \tag{2.5}$$

for $0<|x-u|<\delta$. On the other hand, by the choice of N, we have $|f_m'(y)-f_N'(y)|<\varepsilon$ for all $y\in(a,b)$ and $m\geq N$. So we have $|g(u)-f_N'(u)|\leq\varepsilon$. This together with Eq.2.5 give

$$|\frac{f(x)-f(u)}{x-u}-g(u)|\leq 3\varepsilon$$

as $0<|x-u|<\delta$, that is we have

$$\lim_{x\to u}\frac{f(x)-f(u)}{x-u}=g(u).$$

The proof is finished. □

Remark 2.3. *The uniform convergence assumption of (f_n') in Propositions 2.1 and 2.2 is essential.*

Example 2.4. *Let $f_n(x):=\tan^{-1}nx$ for $x\in(-1,1)$. Then we have*

$$f(x):=\lim_n\tan^{-1}nx=\begin{cases}\pi/2 & \textit{if } x>0;\\ 0 & \textit{if } x=0;\\ -\pi/2 & \textit{if } x<0.\end{cases}$$

Also $g(x):=\lim_n f_n'(x)=\lim_n 1/(1+n^2x^2)=0$ for all $x\in(-1,1)$. So Propositions 2.1 and 2.2 does not hold. Note that (f_n') does not converge uniformly to g on $(-1,1)$.

References

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